Test Functions and Constrained Interpolation

James Pickering

University of Newcastle-upon-Tyne

17th December 2008

The Nevanlinna-Pick Problem

In the classic **Nevanlinna-Pick problem**, we have n points $z_1, \ldots, z_n \in \mathbb{D}$ (the unit disc) and another n points $w_1, \ldots, w_n \in \mathbb{D}$. We want to find a holomorphic function

$$f : \mathbb{D} \longrightarrow \mathbb{D}$$
$$\text{with } f : z_i \longrightarrow w_i$$

Equivalently, we're looking for bounded analytic functions on the disc, with a given norm, that take the prescribed values at the prescribed points.

The Solution

The solution, which was independently discovered by Nevanlinna and Pick, is that a holomorphic function $f : \mathbb{D} \to \mathbb{D}$ with $f(z_i) = w_i$ exists if and only if the matrix

$$\left(\frac{1 - w_i \overline{w_j}}{1 - z_i \overline{z_j}} \right)_{i,j=1}^{n}$$

is positive.

Hardy Spaces

This solution can be interpreted by way of **Hardy spaces**.

Analytic functions on the unit disc $\mathbb{D}$ have (somewhat) well defined values on the boundary of $\mathbb{D}$ – the circle $\mathbb{T}$.

Since there is an obvious measure on $\mathbb{T}$, we can define L^p spaces, $L^p(\mathbb{T})$. The **Hardy space** H^p is the "Holomorphic subspace" of $L^p(\mathbb{T})$.

Multiplicative Action

In this context, bounded analytic functions like f can be thought of as functions in H^∞.

Also, it's easy to show that for any function $f \in H^\infty$ we have a corresponding **multiplication operator**

$$M_f : H^2 \to H^2 \quad M_f(h)(z) := (fh)(z)$$

and that $\|M_f\|_{B(H^2)} = \|f\|_{H^\infty}$.

Reproducing kernels

It's possible to show, by Cauchy's integral theorem, that H^2 is a **Hilbert Function Space** (sometimes called a **reproducing kernel Hilbert space**). This means that for each $x \in \mathbb{D}$ we have a function $k_x \in H^2$ such that

$$\langle h,\ k_x \rangle = h(x)$$

so this k_x acts like a δ-function on H^2. It sometimes makes sense to write $k_x(y)$ as $k(x,\ y)$.

An Operator Theoretic Interpretation

The reproducing kernel for H^2 is

$$k(x,\ y) = (1 - \overline{x}y)^{-1}\ ,$$

which is known as the **Szegő kernel**.

Knowing this, we can see that the Nevanlinna-Pick matrix criterion can be written as:

$$((1 - w_i\overline{w_j})k(z_j,\ z_i))_{i,j=1}^{n} \geq 0$$

Here's yet another way of thinking about this: If we assume, for a moment, that the interpolation problem has a solution f, an equivalent condition is that

$$\left\| P_{\mathfrak{M}_z} M_f |_{\mathfrak{M}_z} \right\| \leq 1$$

where

$$\mathfrak{M}_z := \operatorname{span}\left\{ k_{z_1}, \ \ldots, \ k_{z_n} \right\}$$

So, although we want to know the norm of f – how it acts globally – it's enough to know what its norm is when we localise it to $z_1, \ldots, z_n$ in H^2 (by taking its compression).

Multiply-connected domains

Nevanlinna and Pick considered holomorphic functions on the disc $\mathbb{D}$.

If we have some other planar domain R, we can define $H^\infty(R)$ in much the same way as before.

However, if R is not simply connected, then $H^\infty(R)$ acts by multiplication on many different Hilbert function spaces, not just $H^2(R)$.

The generalisation of Nevanlinna-Pick to multiply connected domains uses some of these other spaces.

Theorem (Abrahamse)

Let R be a g-holed domain in $\mathbb{C}$. There is a collection of Hilbert function spaces $\left\{H^2{}_\gamma : \gamma \in \mathbb{T}^g\right\}$ such that: There is a function $f \in H^\infty(R)$ with $f(z_i) = w_i$ and $\|f\| \leq 1$ if and only if

$$((1 - w_i\overline{w_j})k_\gamma(z_j,\ z_i))_{i,j=1}^n \geq 0$$

for all $\gamma \in \mathbb{T}^g$.

Abrahamse found that it was necessary to use a family of kernels (from a family of spaces) for interpolation in multiply connected domains. The family he used was parameterised by the g-torus.

Equivalently, we could have written our positivity condition as

$$\left\| P_{\mathfrak{M}_z{}^\gamma} M_f|_{\mathfrak{M}_z{}^\gamma} \right\| \leq 1 \ \forall \gamma \in \mathbb{T}^g$$

with

$$\mathfrak{M}_z{}^\gamma := \{ k^\gamma{}_{z_1}, \ \ldots, \ k^\gamma{}_{z_n} \}$$

That is, to establish that f is contractive globally, we need to know that f is contractive when we localise it in **each** of our Hilbert function spaces H^2_γ.

Constrained Interpolation

Davidson et al. looked at what happened when we put some restrictions on the interpolating function, by requiring that $f'(0) = 0$.

Equivalently, they looked at interpolating on a subalgebra of $H^\infty(\mathbb{D})$ with constraints on derivatives, the algebra

$$H_1^\infty = \{f \in H^\infty : \ f'(0) = 0\}$$

As with Abrahamse result for $H^\infty(R)$, H_1^∞ can act on many possible Hilbert function spaces. Davidson at al. found that a family of kernels, parameterised by the unit sphere S^2, was sufficient.

A Different Approach

Another way to rewrite the Nevanlinna-Pick condition is

$$(1 - w_i\overline{w_j}) = \Gamma_{ij}\,(1 - z_i\overline{z_j})$$

For some positive matrix Γ.

This is the same as saying that $(1 - w_i\overline{w_j})_{ij}$ is the Schur product of a positive matrix Γ and $(1 - z_i\overline{z_j})_{ij}$.

In fact, if this holds – so we have a solution f to the interpolation problem – then we know that there is a positive **kernel** $\Gamma : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$, so now over the whole of $\mathbb{D}$, such that

$$\left(1 - f(z)\overline{f(w)}\right) = \Gamma(z,\ w)\ (1 - z\overline{w})$$

for all $z,\ w \in \mathbb{D}$.

This gives us a simple example of a realisation.

Test Functions and Realisations

Roughly following Dritschel and McCullough, a set of test functions is a set Θ, containing contractive functions on a space X, which separate X.

These are **not the same** as test functions in distribution theory.

A set of test functions Θ gives us a normed algebra of functions on X via realisation. We denote this realisation algebra by H^∞_Θ.

We say a function $f : X \to \mathbb{C}$ is in the unit ball of H^∞_Θ, if there's some positive kernel $\Gamma : X \times X \to C_b(\Theta)^*$ such that

$$\left(1 - f(z)\overline{f(w)}\right) = \Gamma(z,\ w)\ \left(1 - E(z)\overline{E(w)}\right)$$

for all $z,\ w \in X$.

Here, $E(z) \in C_b(\Theta)$ is the point evaluation function at z.

If Θ is compact, this formula can be written

$$\left(1 - f(z)\overline{f(w)}\right) = \int_\Theta \Gamma_\vartheta(z,\ w)\ \left(1 - \vartheta(z)\overline{\vartheta(w)}\right) d\vartheta$$

Uses

As before, if we have a realisation at a finite number of points of X, we have a realisation at all points in X.

Also, it can be shown that functions in $H^\infty_\ominus$ can be written as **transfer functions** – structures that come up naturally in control engineering.

Kernels

We can also relate the test function approach back to Abrahamse's kernel based approach.

A set of test functions naturally induces a set of kernels,

$$\mathcal{K}_{\Theta} := \left\{ k : X \times X \to \mathbb{C} : \left(1 - \vartheta(x)\overline{\vartheta(y)}\right) k(x, y) \geq 0 \ \forall \vartheta \in \Theta \right\}$$

Similarly, a set of kernels naturally induces a normed algebra. We say a function $f : X \to \mathbb{C}$ is in the unit ball of $H^\infty_{\mathcal{K}}$ if

$$\left(1 - f(x)\overline{f(y)}\right) k(x,\ y) \geq 0 \ \forall k \in \mathcal{K}$$

In fact, the algebra of functions we get from $\mathcal{K}_\Theta$ is precisely the realisation algebra of Θ, so $H^\infty_{\mathcal{K}_\Theta} = H^\infty_\Theta$. This means we can use these kernels to solve interpolation problems on H^∞_Θ.

The Bidisc

The original motivation for this approach was the bidisc (that is, $\mathbb{D} \times \mathbb{D}$).

Test functions are a particularly natural way of describing the kernels needed for interpolation on the bidisc – interpolation in $H^\infty(\mathbb{D}^2)$ needs a lot of kernels, but only two test functions – and first appeared in connection with this problem.

Representations

Test functions are also relevant when we're searching for contractive representations for our algebra H^∞_Θ; we know that "well behaved" representations of H^∞_Θ are contractive provided they're contractive on Θ.

This can be useful when looking at completely contractive representations; test functions are a useful tool when working on rational dilation problems.

Minimality

We typically want to find a set of test functions Θ such that H^∞_Θ is isometrically isomorphic to another algebra, such as $H^\infty(X)$ for some space X.

In these cases, we can simply take Θ to be the unit ball of $H^\infty(X)$, although it simplifies calculations if we take as small a set as possible. Our goal is to find a minimal set of test functions for a given algebra.

Herglotz Representation

We want to find a set of test functions for H^∞_1.

Test functions take values in the unit disc $\mathbb{D}$. If we apply a Cayley transformation, our functions now take values in the right half plane $\mathbb{H}$. This allows us to apply the **Herglotz representation theorem**.

Theorem (Herglotz)

If $f : \mathbb{D} \to \mathbb{H}$ is a holomorphic function with $f(0) > 0$, then there is some positive measure μ on $\mathbb{T}$ such that

$$f(z) = \int_{\mathbb{T}} \frac{w+z}{w-z} d\mu(w)$$

Zero-Mean Measures

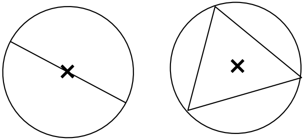
We want our test functions to be in H_1^∞, which means they need to have zero derivative at zero. We can relate this to the measure μ; we can show that $f'(0) = 0$ if and only if

$$\int_{\mathbb{T}} w d\mu(w) = 0$$

If we normalise μ to be a probability measure, this says that $\mathbb{E}(\mu) = 0$ – the expectation of μ is zero. We'll say that μ is a **zero-mean** probability measure if this is the case.

Extreme Measures

We're trying to find a **minimal** set of test functions, so we'd like to find some simple, building-block measures – ideally the simplest possible – that we can build other measures from.



Agler-Herglotz Representation

These "simplest" measures, are also "extreme points", in the sense of Krein-Milman, so **all zero-mean measures** can be built from these extreme measures, $\vartheta \in \Theta$. This gives us a Herglotz-ish representation, which I've called an **Agler-Herglotz representation**.

Theorem (Agler-Herglotz representation)

If $f : \mathbb{D} \to \mathbb{H}$ is a holomorphic function with $f(0) > 0$ and $f'(0) = 0$, then there is some positive measure ν on Θ such that

$$f(z) = \int_\Theta h_\vartheta(z) d\nu(\vartheta)$$

Test Functions

It's fairly simple to turn an Agler-Herglotz representation (parameterised by a set Θ) into a set of test functions (also parameterised by Θ).

In the case of H_1^∞, once we eliminate some redundant test functions from Θ, we're left with a minimal set of test functions, which is parameterised by S^2, the sphere.

An Intriguing Connection

- Recall, Davidson et al. found that the kernels for H_1^∞ were parameterised by the unit sphere S^2. We found that the test functions were parameterised by S^2.
- Abrahamse found that the kernels for $H^\infty(R)$ were parameterised by $\mathbb{T}^g$. Dritschel, McCullough and P. found that the test functions were also parameterised by $\mathbb{T}^g$.
- A possible counterexample to this pattern is the bidisc, $\mathbb{D}^2$. It's known that $H^\infty(\mathbb{D}^2)$ only needs two test functions, but work by McCullough, Paulsen and Solazzo suggests it needs infinitely many kernels.

Further Reading

- My work: `http://www.jamespic.me.uk`
- **A Constrained Nevanlinna-Pick Interpolation Problem**, Davidson, Paulsen, Raghupathi and Singh
- **Test Functions, Kernels, Realizations and Interpolation**, Dritschel and McCullough
- **Pick Interpolation and Hilbert Function Spaces**, Agler and McCarthy