Examples and Applications of Test Functions

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Introduction

Test functions provide a useful framework for working on function theory problems.

Many function theory problems can be approached using test functions.

When test functions can be used, they allow us to use various existing tools, such as **kernels**, **functional models**, **transfer functions** and **normed representations**.

We will introduce the test function framework, and give some examples that demonstrate the capabilities and limitations of the test function approach.

Nevanlinna-Pick Interpolation

Nevanlinna and Pick independently solved the bounded holomorphic interpolation problem:

Theorem

There exists a holomorphic function $f : \mathbb{D} \to \mathbb{D}$ with $f(z_i) = w_i$ for $i = 1, \ldots, n$, if and only if the matrix

$$\left(\frac{1 - w_i \overline{w_j}}{1 - z_i \overline{z_j}} \right)_{i,j} \geq 0$$

We could equivalently write that $\|f\|_{H^\infty} \leq 1$, and that $(1 - w_i \overline{w_j})\, k(z_i, z_j) \geq 0$, where k is the Szegő kernel.

Multiplication

If we think about f as a function in H^∞, then M_f is a multiplication operator on H^2. Recall that k is the reproducing kernel for H^2.

In terms of kernels and multiplication operators, we know that $\|M_f\|_{B(H^2)} \le 1$, precisely when

$$\left(1 - f(z_i)\overline{f(z_j)}\right) k(z_i, z_j) \ge 0$$

for all finite subsets $\{z_1, \ldots, z_n\} \subseteq \mathbb{D}$.

This connection between functions (and their multiplication operators), and reproducing kernels (and their Hilbert spaces), gives us a sort of duality.

Kernel-Function Duality

If we have a collection of functions Ψ, there is a dual collection of reproducing kernels:

$$\mathcal{K}_\Psi = \left\{k : \left(1 - f(x)\overline{f(y)}\right) k(x, y) \geq 0 \, \forall f \in \Psi\right\}$$

Conversely, for any collection of reproducing kernels $\mathcal{K}$, there is a dual collection of functions:

$$\mathcal{B}H^\infty{}_\mathcal{K} = \left\{f : \left(1 - f(x)\overline{f(y)}\right) k(x, y) \geq 0 \, \forall k \in \mathcal{K}\right\}$$

Typically, this forms the unit ball of a Banach algebra, $H^\infty{}_\mathcal{K}$

As we might expect, the double dual of a collection of functions is its closure or span, in some sense.

Test Functions

If we have a collection of functions Ψ, which are strictly contractive on their domain (which we usually write X), and which separate X, then we call Ψ a **collection of test functions**.

Its "double dual" $H^\infty{}_{\mathcal{K}_\Psi}$ is automatically a Banach algebra, and we say Ψ generates $H^\infty{}_{\mathcal{K}_\Psi}$.

Our aim is generally to find a collection of test functions that generates a particular Banach algebra, ideally with as few generators as possible.

For simplicity, I will additionally (and unusually) assume that Ψ is compact.

An example

Jim Agler gave a typical example of a good collection of test functions.

Theorem

The set $\{z_1, z_2\}$ *generates* $H^\infty(\mathbb{D}^2)$.

As we will see later, $\{z_1, \ldots, z_n\}$ does not generally generate $H^\infty(\mathbb{D}^n)$. In fact, it generates a strict subset of $H^\infty(\mathbb{D}^n)$.

Theorem

The following are equivalent:

- $f \in H^\infty{}_{\mathcal{K}_\Psi}$ *and* $\|f\|_{H^\infty{}_{\mathcal{K}_\Psi}} \leq 1$
- *There exists a positive kernel* k_ψ *on* X*, and a measure* μ *on* Ψ *such that*

$$1 - f(x)\overline{f(y)} = \int_\Psi k_\psi(x, y)\left(1 - \psi(x)\overline{\psi(y)}\right) d\mu(\psi)$$

- *For every finite subset* $\mathbf{F} \subset X$*, there exists a positive kernel* k_ψ *on* $\mathbf{F}$*, and a measure* μ *such that*

$$1 - f(x)\overline{f(y)} = \int_\Psi k_\psi(x, y)\left(1 - \psi(x)\overline{\psi(y)}\right) d\mu(\psi)$$

for all $x, y \in \mathbf{F}$

Theorem

The following are equivalent:

- $f \in H^\infty{}_{\mathcal{K}_\Psi}$ *and* $\|f\|_{H^\infty{}_{\mathcal{K}_\Psi}} \leq 1$
- *There is a representation* $\rho : C_b(\Psi) \to B(H)$ *and a unitary* $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ *on* $H \oplus \mathbb{C}$ *such that*
$$f(z) = D + C\rho(\widehat{z})\,(I - A\rho(\widehat{z}))^{-1}\,B$$
- *If* $\pi : H^\infty{}_{\mathcal{K}_\Psi} \to B(H)$ *is a representation such that* $\|\pi(\psi)\| < 1$ *for all* $\psi \in \Psi$*, then* $\|\pi(f)\| \leq 1$
- *If* $\pi : H^\infty{}_{\mathcal{K}_\Psi} \to B(H)$ *is a weakly continuous representation such that* $\|\pi(\psi)\| \leq 1$ *for all* $\psi \in \Psi$*, then* $\|\pi(f)\| \leq 1$

Interpolation

The motivating problem for these techniques is interpolation.

Theorem

The following are equivalent:

- $\exists$ *a function* $\zeta \in H^{\infty}{}_{\mathcal{K}_\Psi}$ *with* $\|f\| \leq 1$ *and* $f(z_i) = w_i$.
- *For each* $k \in \mathcal{K}_\Psi$

$$\left(1 - w_i\overline{w_j}\right) k(z_i, z_j) \geq 0$$

- $\exists$ *a positive kernel* k_ψ *and measure* μ *such that*

$$1 - w_i\overline{w_j} = \int_\Psi k_\psi(z_i, z_j) \left(1 - \psi(z_i)\overline{\psi(z_j)}\right) d\mu(\psi)$$

The Bidisc

In the bidisc, our positive kernel condition is precisely Agler's condition for interpolation on the bidisc:

$$1 - w_i \overline{w_j} = \Gamma_1(z_i, z_j)\left(1 - z_i^1 \overline{z_j^1}\right) + \Gamma_2(z_i, z_j)\left(1 - z_i^2 \overline{z_j^2}\right)$$

Some authors call this a **model** for the function f

The Tridisc and Beyond

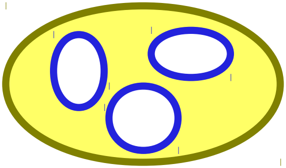
In light of the main theorem, we can see the problem with the tridisc and above (recall $\Psi = \{z_1, \ldots, z_n\}$).

We know that there is no von Neumann type inequality in the tridisc, so our normed representation condition must fail for some contractive functions.

The contractive functions that satisfy the von Neumann inequality (or any of the equivalent conditions) are often called the **Schur-Agler class**.

Despite the fact that we can't represent the tridisc in this way (so $H^\infty(\mathbb{D}^3) \neq H^\infty{}_{\mathcal{K}_\Psi}$), $H^\infty{}_{\mathcal{K}_\Psi}$ still forms a well defined Banach algebra

Test functions in $H^\infty(X)$



Dritschel and McCullough gave a collection of test functions that generates $H^\infty(X)$, when $X \subseteq \mathbb{C}$. Their collection is parameterised by $\mathbb{T}^n$.

It consists of inner functions on X, that are 1–1 on the boundary components. $\mathbb{T}^n$ describes (roughly) how out of phase the boundaries are.

Constrained Interpolation

Another interesting interpolation was introduced by Davidson, Paulsen, Raghupathi and Singh.

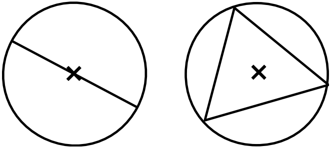
They looked at interpolation in the algebra

$$H_1^\infty := \{f \in H^\infty(\mathbb{D}) : f'(0) = 0\}$$

They showed that interpolation in this algebra could be checked with a collection of kernels.

This is similar to Abrahamse solution to interpolation on $H^\infty(X)$, or condition 2 of our interpolation theorem.

Test functions in H_1^∞



We have a collection of test functions for H_1^∞.

It consists of inner functions on $\mathbb{D}$, which are equal to 1 at certain points on the boundary – either:

- three points, which form a triangle with 0 in its interior, or
- two points, on opposite sides of the circle.

This is an oversimplification – details were presented at IWOTA 2009.

Finite Collections

It would seem natural to ask if we could get away with fewer test functions.

For example, can we generate $H^\infty(\mathbb{A})$ using $\Psi = \{z, \frac{r}{z}\}$?

The collection of test functions we gave before was shown to be minimal (i.e, has no subset that generates $H^\infty(\mathbb{A})$) by Dritschel and McCullough, but this new Ψ is not a subset of it, so could it be sufficient?

Continuous embeddings

Using the main theorem, we can relate this to K-spectral sets.

By a result of Douglas and Paulsen, we can show that Ψ generates the **algebra** $H^\infty(\mathbb{A})$, but with a different **norm**.

In fact, the identity mapping between $H^\infty(\mathbb{A})$ and $H^\infty{}_{\mathcal{K}_\Psi}$ is a continuous (but not isometric) isomorphism.

The interpolation theorem can help us to interpret this.

Interpolation, and The Bidisc

We can embed $\mathbb{A}$ in $\mathbb{D}^2$ as

$$\mathbb{A} \cong \left\{ \left(z, \frac{r}{z} \right) : z \in \mathbb{A} \right\} \subseteq \mathbb{D}^2$$

By a variant of the interpolation theorem, we can extend any function f in $H^\infty{}_{\mathcal{K}_\Psi}$ to the whole of $\mathbb{D}^2$.

If we find the extension of f with the lowest norm in $H^\infty(\mathbb{D}^2)$, then this is precisely the norm of f in $H^\infty{}_{\mathcal{K}_\Psi}$.

Essentially the same result holds for other planar domains.

In the sense above, the bidisc is the **universal model** for pairs of test functions.

$$\{(\psi_1(z), \psi_2(z)) : z \in X\} \subseteq \mathbb{D}^2$$

A similar principle applies to larger finite collections of test functions. However, we are now embedding in $H^\infty{}_{\mathcal{K}_{\{z_1,\ldots,z_n\}}}$, which need not be the same as $H^\infty(\mathbb{D}^n)$.

Nonetheless, as with the annulus, if X is a planar domain with circular boundaries, and the obvious finite collection of test functions, then $H^\infty(X)$ is again continuously isomorphic to $H^\infty{}_{\mathcal{K}_{\{\psi_1,\ldots,\psi_n\}}}$

We can do the same thing in H^∞_1, using the test functions

$$\Psi = \left\{z^2, z^3\right\}$$

In this case, this is a strict subset of our previous collection of test functions, so can't generate the same normed algebra.

However, by a result of Polyakov, the algebra generated is, once again, continuously isomorphic to H^∞_1.

Polyakov's result looks at the embedding $(z^2, z^3) \subseteq \mathbb{D}^2$ directly.

Summary

- Using function-kernel duality, we can talk about functions generating an algebra.
- When we have a generating set for an algebra, we automatically have functional models, transfer function representation, norm inequalities, and interpolation.
- This approach works well for the disc, the bidisc, planar domains, and some constrained algebras.
- In three or more dimensions, the scope of this approach is more limited.
- In some cases, smaller, finite collections of test functions can work, but give us less control over norms. Norms are inherited from the polydisc, which acts as a universal model.

Further Reading

- **My work**: `http://www.jamespic.me.uk`
- **Pick Interpolation and Hilbert Function Spaces**: Agler and McCarthy
- **Test functions, kernels, realizations and interpolation**: Dritschel and McCullough