EXAMPLES AND APPLICATIONS OF TEST FUNCTIONS (HANDOUT)

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To help you remember, here are some ideas and results that we'll need to use later on in this talk. I'll explain through them in the talk, but this handout should help keep them fresh in your mind. The talk itself is available on my website, `http://www.jamespic.me.uk`.

Firstly, there's the function-kernel duality. If we start from the right functions (so called test functions), this duality can "complete" our functions, giving a Banach algebra.

Definition. If we have a collection of functions Ψ, there is a dual collection of kernels:

$$\mathcal{K}_\Psi = \left\{k : \left(1 - f(x)\overline{f(y)}\right) k(x,\, y) \geq 0\, \forall f \in \Psi\right\}$$

Conversely, for any collection of kernels $\mathcal{K}$, there is a dual collection of functions:

$$\mathcal{B}H^\infty_{\mathcal{K}} = \left\{f : \left(1 - f(x)\overline{f(y)}\right) k(x,\, y) \geq 0\, \forall k \in \mathcal{K}\right\}$$

We say that a collection Ψ of functions on a space X is a set of test functions, if

- $|\psi(x)| < 1$ for all $x \in X$, and
- Ψ separates points in X, in the sense that, if we restrict functions from Ψ to finite subsets $F \subseteq X$, then the unital algebra they generate is the algebra of all functions on F.

We say that Ψ generates $H^\infty_{\mathcal{K}_\Psi}$ – its "double dual".

We also need some notation, in order to state the main result in this field.

Definition. If Ψ is a collection of test functions, then:

- $C_b(\Psi)$ denotes the bounded, continuous functions on Ψ.
- $C_b(\Psi)^*$ denotes its bounded dual space (if Ψ is compact, this would be all borel measures on Ψ).
- If $x \in X$, then $\widehat{x} \in C_b(\Psi)$ is the evaluation functional, $\widehat{x}(f) = f(x)$.

The main result in this field is the following, which allows us to use a number of existing tools from function theory

Theorem. *The following are equivalent:*

(1) *$f \in H^\infty_{\mathcal{K}_\Psi}$ and $\|f\|_{H^\infty_{\mathcal{K}_\Psi}} \leq 1$*

(2) *There exists a positive kernel k_ψ on X, and a measure μ on Ψ such that*

$$1 - f(x)\overline{f(y)} = \int_\Psi k_\psi(x,\, y)\left(1 - \psi(x)\overline{\psi(y)}\right) d\mu(\psi)$$

(3) *For every finite subset $\mathbf{F} \subset X$, there exists a positive kernel k_ψ on $\mathbf{F}$, and a measure μ such that*

$$1 - f(x)\overline{f(y)} = \int_\Psi k_\psi(x,\, y)\left(1 - \psi(x)\overline{\psi(y)}\right) d\mu(\psi)$$

(4) *There is a representation $\rho : C_b(\Psi) \to B(H)$ and a unitary $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ on $H \oplus \mathbb{C}$ such that*

$$f(z) = D + C\rho(\widehat{z})\left(I - A\rho(\widehat{z})\right)^{-1} B$$

(5) *If $\pi : H^\infty{}_{\mathcal{K}_\Psi} \to B(H)$ is a representation such that $\|\pi(\psi)\| < 1$ for all $\psi \in \Psi$, then $\|\pi(f)\| \leq 1$*

(6) *If $\pi : H^\infty{}_{\mathcal{K}_\Psi} \to B(H)$ is a weakly continuous representation such that $\|\pi(\psi)\| \leq 1$ for all $\psi \in \Psi$, then $\|\pi(f)\| \leq 1$*

The most obvious consequence of this is an interpolation theorem, for test function algebras.

Theorem. *The following are equivalent:*

(1) *There exists a function $\zeta \in H^\infty_{\mathcal{K}_\Psi}$ with $\|f\| \leq 1$ and $f(z_i) = w_i$.*

(2) *For each $k \in \mathcal{K}_\Psi$*

$$\left(1 - w_i\overline{w_j}\right) k(z_i,\, z_j) \geq 0$$

(3) *There exists a positive kernel k_ψ and measure μ such that*

$$1 - w_i\overline{w_j} = \int_\Psi k_\psi(z_i,\, z_j)\left(1 - \psi(z_i)\overline{\psi(z_j)}\right) d\mu(\psi)$$

Further Reading.

- *My work*: `http://www.jamespic.me.uk`
- *Pick Interpolation and Hilbert Function Spaces*: Agler and McCarthy
- *Test functions, kernels, realizations and interpolation*: Dritschel and McCullough